Homework Set 5

Due Date Thursday, October 28, 1999, at the beginning of class. Hint: the homework is intended to be completed over **two** weeks; do not leave it until midnight of Wednesday, October 27, 1999!

1 Binary Formation in Globular Clusters

Binaries are key elements of a globular cluster because, as discussed in lectures, they store a large amount of gravitational binding energy. When this energy is released, it significantly affects the evolution of the cluster.

Some of the binaries in a globular cluster are primordial, i.e. they formed at the same time as the cluster. Other binaries are created later through stellar encounters of different kinds. In this problem, we look at two creation mechanisms: three-body encounters, and tidal capture.

(Hint: many of the calculations in this question are order-of-magnitude; do not be afraid to approximate when you cannot calculate something exactly.)

(A) Three-Body Encounters. Three stars approach close enough together to capture each other gravitationally and form a binary or triple system, as in Fig. 1. We assume the stars are point masses, so that there are zero tidal losses.

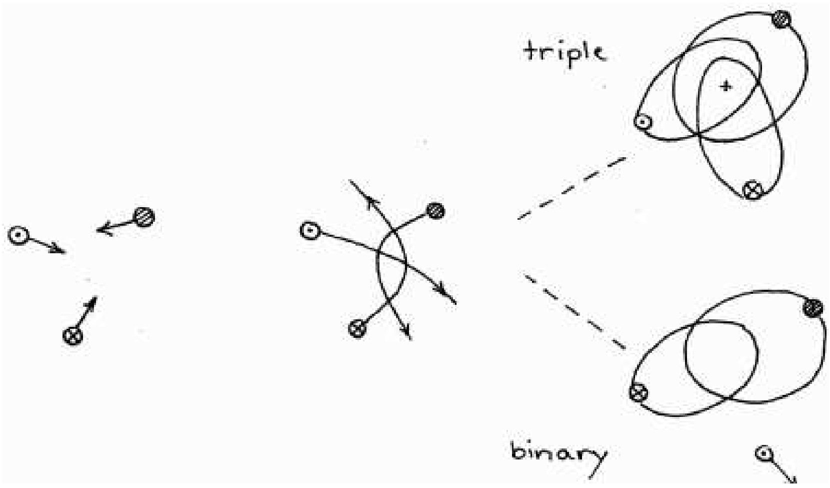

Figure 1:

1. Explain why two point masses that are unbound initially cannot become bound after an encounter unless a third body is simultaneously nearby.

2. Suppose all three stars have mass m and speed v. Show that the velocity perturbation of each star is given by $\Delta v \approx Gm/bv$, where b is the approximate distance of closest approach of the stars. (Hint: take an order-of-magnitude approach.)

3. Use the result 2. to argue that the condition $b \approx Gm/v^2$ must hold if two or more of the stars are to become bound after the encounter. Explain your reasoning carefully.

4. Compare $b \approx Gm/v^2$ to the characteristic (i) radius and (ii) separation of stars in a globular cluster. Use typical numbers such as those in the lecture notes. Comment on what you find.

5. Let n be the number of stars per unit volume in the cluster. Obtain an expression, in terms of n, b, and v, for how long it takes before **two** stars approach within a distance b of each other.

6. What is the probability, in terms of n and b, that a **third** star will be in the vicinity when the other two stars meet?

7. Using your answers to 3., 5., and 6., show that the time between triple encounters is roughly $t_{\rm triple} \approx v^9/n^2G^5m^2$.

8. Evaluate $t_{\rm triple}$ numerically for typical globular cluster parameters (i) in the core and (ii) in the halo. Is binary creation by three-body encounters a significant process in globular clusters?

(B) Tidal Capture. Two stars, initially unbound, approach close together, each raising a tidal bulge on the other. As the tidal bulges change orientation during the encounter, orbital energy is dissipated as heat and the two stars bind to form a binary, as in Fig. 2. In this case, the stars are not point masses.

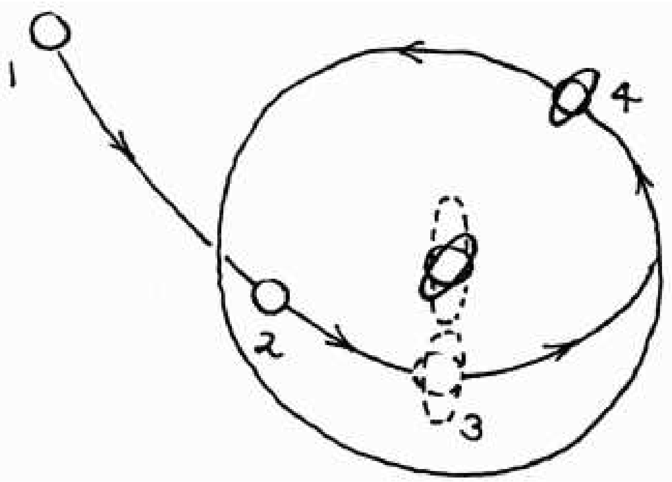

Figure 2:

9. Explain why two stars that are unbound initially can become bound after an encounter if tidal losses are present. Compare with the situation in 1.

10. A tidal capture binary is formed as a result of a close encounter of two stars of equal mass m. The minimum separation during the encounter is $b_{\rm min}$, and the orbital energy dissipated in the encounter satisfies $\Delta E \ll Gm^2/b_{\rm min}$. After the binary has formed, with an elliptical orbit in general, more energy is dissipated in each successive orbit until the orbit is circularised. Ignoring the spin of the two stars completely, show that the radius of the final circular orbit is $2b_{\rm min}$. (Hint: this part has nothing to do with any of the previous parts; think afresh.)

2 Core Collapse by Evaporation

Evaporation of stars from a globular cluster causes the cluster to shrink over time.

1. Write a paragraph explaining how this happens physically. Start from first principles and include **every step** in your reasoning. At the same time, be **concise** ($\leq 2/3$ page including any diagrams); aim for clarity through economy of expression. (This exercise is intended to help you practise writing 'essay paragraphs' in exams.)

2. Suppose evaporation occurs sufficiently slowly for the cluster to remain in equilibrium at every stage of the collapse. Using the virial theorem, express the total mechanical energy of the cluster, $E(t)$, in terms of its mass $M(t)$ and radius $R(t)$. Your answer will be approximate; ignore factors of order unity.

3. Each evaporating star carries away very little total mechanical energy as it escapes the cluster. Explain why, physically, this is so.

4. Use 2. and 3. to show that the rate of change of the cluster radius is related to the mass loss rate by
$$\frac{dR(t)}{dt} = \frac{2R(t)}{M(t)}\frac{dM(t)}{dt} \,. \tag{1}$$

5. Also use 2. and 3. to relate $M(t)$ and $R(t)$ to the initial mass M_0 and radius R_0 by
$$\frac{M(t)^2}{M_0^2} = \frac{R(t)}{R_0} \tag{2}$$

The mass loss rate is given approximately by
$$\frac{dM(t)}{dt} \approx -\frac{M(t)}{\tau_{\rm evap}(t)} \,, \tag{3}$$

where $\tau_{\rm evap}(t) = k\tau_{\rm relax}(t)$, with $k = 3 \times 10^2$, is the evaporation time. The relaxation time $\tau_{\rm relax}(t)$ is given by the Spitzer-Hart formula (see lecture notes)
$$\tau_{\rm relax}(t) = \frac{0.14M(t)/m}{\ln \Lambda(t)} \left[\frac{R(t)^3}{GM(t)}\right]^{1/2} \tag{4}$$

with $\Lambda(t) = 0.4M(t)/m$, provided that all stars in the cluster have the same mass m.

6. Why does $\tau_{\rm relax}(t)$ change with time t as the cluster evolves? Why is k roughly constant? Please give physical, not mathematical, explanations. Be precise.

7. Combine (1)–(4) to eliminate $M(t)$ and obtain a first-order differential equation for how $R(t)$ evolves with time.

8. Assume $\ln\Lambda(t)$ is constant with time; logarithms change very slowly. Integrate the equation in 7. and show that $R(t)$ is of the form

$$R(t) = R_0(1 - t/t_{\rm c})^{4/7} \,. \tag{5}$$

   Express $t_{\rm c}$ in terms of G, m, M_0, and R_0. Sketch $R(t)$ as a function of t.

9. Evaluate numerically how long it takes for a typical globular cluster to collapse to a point. Then repeat your estimate for just the core of a cluster. Use typical numbers such as those in the lecture notes. Comment.

10. Apply the virial theorem to obtain $V(t)$, the characteristic stellar speed, in terms of $R(t)$ and hence G, M_0, R_0 and $t_{\rm c}$. Sketch $V(t)$ as a function of t. What is the value of $V(t_{\rm c})$? Is this realistic?

In reality, a globular cluster does not collapse to a point. Its end state is a single close binary with, say, $R(t_{\rm f}) = 1\,{\rm AU}$ and $V(t_{\rm f}) = 40\,{\rm km\,s^{-1}}$, as indicated by numerical simulations.

11. Using 8., 10., and $R(t_{\rm f}) = 1\,{\rm AU}$, evaluate $V(t_{\rm f})$ numerically. You should get a much larger speed than $40\,{\rm km\,s^{-1}}$. Can you suggest **two** reasons why the simple theory of this problem does not predict the end state of core collapse correctly?

3 Measuring the Galactic Rotation Curve

Fig. 3 shows the distribution of neutral hydrogen maxima (shaded curves) in the spiral arms of our Galaxy. The position of the Sun is marked by the symbol $\odot$ and its circular orbit is plotted as a dashed curve.

1. Draw the 21-cm line profiles (i.e. 21-cm intensity as a function of frequency ν) you would expect to observe in the directions $l = 50°$, $l = 110°$, and $l = 230°$. Label the points on these profiles corresponding to spiral arms. Do not undertake any detailed velocity calculations, but make sure that the profiles are **qualitatively** accurate and display all the necessary features in correct positions.

The following maximum velocities (relative to the Local Standard of Rest) of neutral hydrogen gas are observed by measuring 21-cm emission along certain lines-of-sight: (i) $123\,{\rm km\,s^{-1}}$, $l = 15°$; (ii) $95\,{\rm km\,s^{-1}}$, $l = 30°$; (iii) $64\,{\rm km\,s^{-1}}$, $l = 45°$; (iv) $29\,{\rm km\,s^{-1}}$, $l = 60°$; (iv) $7.5\,{\rm km\,s^{-1}}$, $l = 75°$.

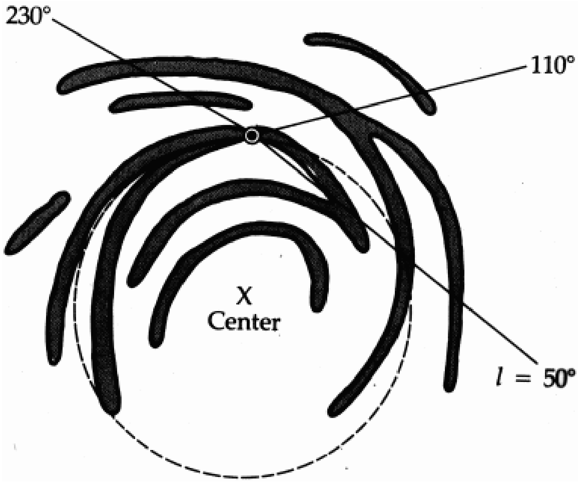

Figure 3:

2. Compute and then plot the rotation curve of the Galaxy interior to the Sun ($R < R_0$) using these data. Remember, the rotation curve is a graph of circular speed $v_c(R)$ as a function of Galactic radius R.

3. Assuming that the mass distribution is spherically symmetric for $R < R_0$, compute and then plot a graph of the mass $M(R)$ enclosed within a radius R, as a function of $R < R_0$.

4. Evaluate numerically the total mass interior to the Sun. Express your answer in solar masses.

5. Compute and then plot a **rough** curve of mass density $\rho(R)$ as a function of $R < R_0$.

4 Velocity of Barnard's Star

(Quick question.) Barnard's star has radial speed $-108\,\mathrm{km\,s^{-1}}$, proper motion $10.34''\,\mathrm{yr}^{-1}$, and parallax $0.546''$. Parallax is the angle by which a star shifts against its background when measured from the two extreme points of a baseline of 1 AU, i.e. observing the star from Earth three months apart.

1. What is the distance to Barnard's star in pc?

2. What is the tangential speed of Barnard's star?

3. What is the magnitude and direction of the velocity of Barnard's star? Express the direction as an angle with respect to the line-of-sight.

4. How far, in pc, will Barnard's star be from Earth at its point of closest approach?

5. In how many years will it reach its point of closest approach?

A piece of trivia: before the recent discovery of extrasolar planets using the Doppler-recoil technique, Barnard's star was a favourite target of astrometric planet searches because of its proximity to Earth. (An astrometric search looks for a wobble in the path of the star across the sky.) Several of these searches even claimed success, though without confirmation. It is now known, from Doppler-recoil, that Barnard's star does not have planets — at least, not massive ones like Jupiter.