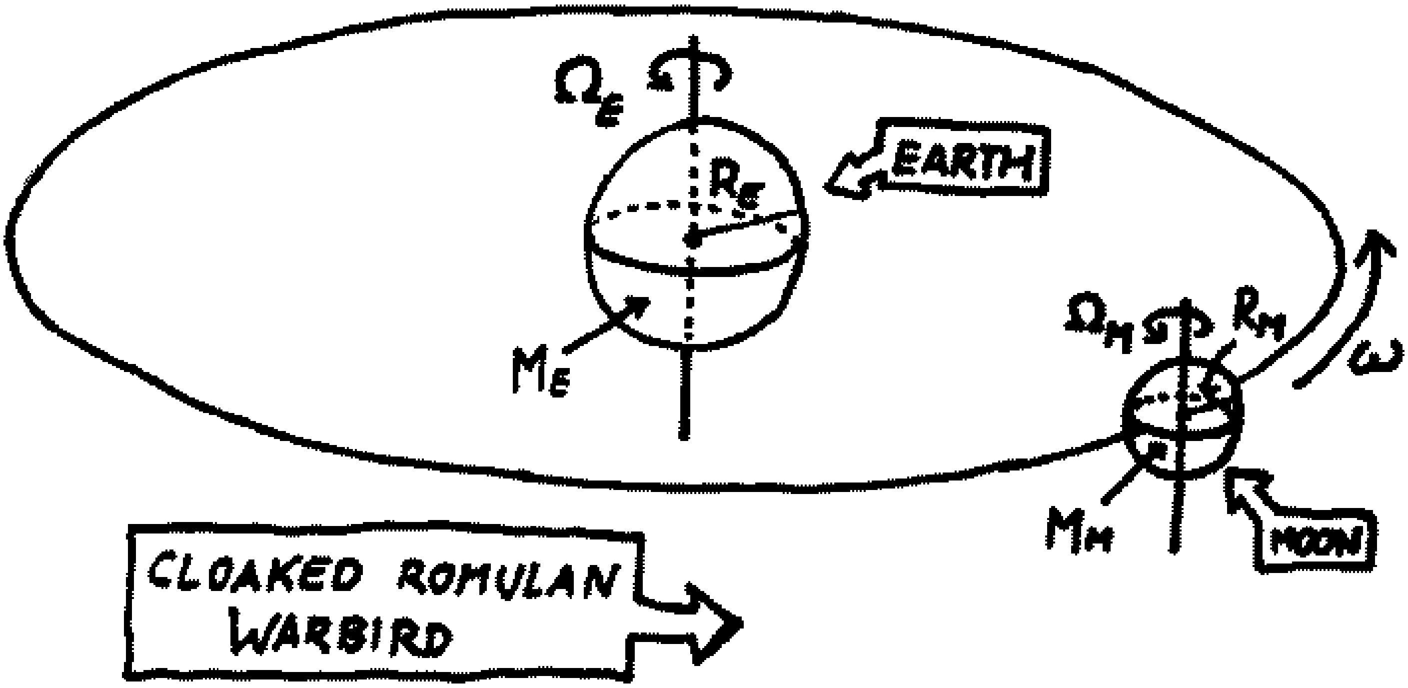
Homework Set 2

Due Date Thursday, September 16, 1999, at the beginning of class.

1 Lengthening of the Terrestrial Day

Consider the Earth-Moon system in its current configuration. The Moon is rotating synchronously, the Earth is rotating asynchronously, and the orbit of the Moon about the Earth is circular.



To answer this question, you will need the following piece of information: a uniform sphere of mass M and radius R, rotating with angular frequency Ω, has moment of inertia $I = \frac{2}{5}MR^2$, spin angular momentum $I\Omega$, and spin kinetic energy $\frac{1}{2}I\Omega^2$. (You may have seen this in your physics courses.)

1. Show that the total angular momentum L and total energy E of the Earth-Moon system are given by

$$L = I_{\rm E}\Omega_{\rm E} + I_{\rm M}\omega + M_{\rm M}(GM_{\rm E})^{2/3}\omega^{-1/3} , \tag{1}$$

$$E = \frac{1}{2}I_{\rm E}\Omega_{\rm E}^2 + \frac{1}{2}I_{\rm M}\omega^2 - \frac{1}{2}M_{\rm M}(GM_{\rm E})^{2/3}\omega^{2/3} , \tag{2}$$

   where subscripts 'E' and 'M' denote Earth and Moon respectively, and ω is the orbital angular frequency.

2. By explicit evaluation, show that the second term is smaller than the first in the right-hand side of each expression. We ignore both small terms in everything that follows.

Let us start by looking at how the Earth-Moon system is currently evolving.

3. Ignoring both small terms, carefully differentiate the angular momentum and energy equations with respect to time. You should find

$$0 = I_{\mathrm{E}} \frac{d\Omega_{\mathrm{E}}}{dt} - \frac{1}{3} M_{\mathrm{M}} (G M_{\mathrm{E}})^{2/3} \omega^{-4/3} \frac{d\omega}{dt} \,, \tag{3}$$

$$\frac{dE}{dt} = I_{\mathrm{E}} \Omega_{\mathrm{E}} \frac{d\Omega_{\mathrm{E}}}{dt} - \frac{1}{3} M_{\mathrm{M}} (G M_{\mathrm{E}})^{2/3} \omega^{-1/3} \frac{d\omega}{dt} \,. \tag{4}$$

   Physically, why is dE/dt not zero? Do you expect it to be positive or negative?

4. Earth's day is currently lengthening at a rate of $20\,\mu$s per year. What is $d\Omega_{\mathrm{E}}/dt$?

5. What is $d\omega/dt$? How fast is the radius of the Moon's orbit changing? (Express your answer in cm per year.) Is the radius increasing or decreasing? Explain your result physically.

6. At what rate is energy being dissipated in the Earth as heat?

7. Compare your answer to 6. with the rate at which solar energy is absorbed by the Earth. Assume that none of the solar energy incident on Earth is reflected. The Sun's luminosity is 3.83×10^{26} W.

Let us now consider the end state of the Earth-Moon system, where the Earth as well as the Moon are synchronously rotating.

8. Using (1), write down the total angular momentum of the system in its current state. Do the same for the end state.

9. Use your answers to calculate how long the Earth's day will be when the end state is reached. How long will a month last?

10. What will be the radius of the Moon's orbit?

11. How much energy will have been dissipated as heat in the course of reaching the end state?

12. What extra piece of information do we need to determine how long it will take the Earth-Moon system to reach its end state?