Homework Set 1

Due Date:
Thursday, September 9, 1999, at the beginning of class. A word of caution: you have two weeks but it is a long homework set. Please do not leave it until the last minute.

1 Launching Satellites

A rocket is launched vertically from the Earth's surface. Its mass (including payload) before fuelling is $M_{\rm r}$; the mass of its fuel is $M_{\rm f}$. Supppose that the rocket burns and expels fuel at a constant rate $\dot{M}_{\rm f}$ (units: $\mathrm{kg\,s^{-1}}$) with a constant exhaust speed $V_{\rm e}$ **in the rocket frame**.

1. What forces act on the rocket **in the frame of the Earth**? Think carefully.
2. Write an expression for the total mass of the rocket, $m(t)$, as a function of time t during the burn phase.

Assume that, during the burn phase, the altitude of the rocket is much less than the radius of the Earth, $R_\oplus$, so that the acceleration due to gravity, g, is nearly constant with altitude.

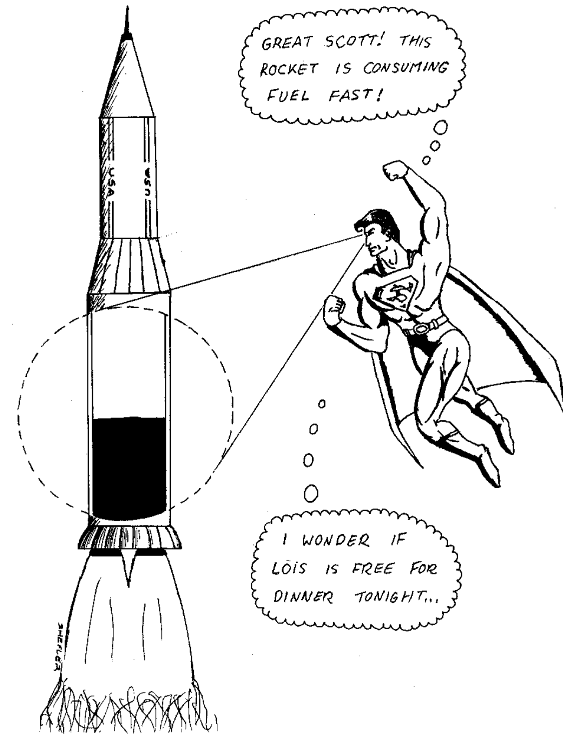
3. Write down Newton's equation of motion for the rocket. Solve it to show that the speed of the rocket, $v(t)$, is given by

$$v(t) = V_{\rm e} \ln \left(\frac{M_{\rm r} + M_{\rm f}}{M_{\rm r} + M_{\rm f} - \dot{M}_{\rm f} t} \right) - gt \tag{1}$$

   during the burn phase.

Take $V_{\rm e} = 3\,\mathrm{km\,s^{-1}}$ and a burn time of 30 s.

4. Argue that $V_{\rm e} = 3\,\mathrm{km\,s^{-1}}$ is a reasonable ballpark estimate of the exhaust speed for a rocket burning a fuel mixture of hydrogen and oxygen. (You will need to look up a piece of chemical data for this question. Hint: $2\mathrm{H}_{2\,(g)} + \mathrm{O}_{2\,(g)} \rightarrow 2\mathrm{H_2O}_{\,(g)} + \text{heat}$.)
5. What is $M_{\rm r}/M_{\rm f}$ if the rocket reaches a speed of $7.7\,\mathrm{km\,s^{-1}}$ at the end of the burn phase? ($7.7\,\mathrm{km\,s^{-1}}$ is the speed of a satellite in low-Earth orbit at an altitude of 300 km, if you are curious!)
6. What is $M_{\rm r}/M_{\rm f}$ if the rocket reaches escape speed at the end of the burn phase? Comment on the practical importance of your answer.
7. What is the altitude of a geostationary satellite?



2 Orbits of a Simple Harmonic Oscillator

Consider a body, of mass m, attached to one end of a spring with spring constant $k = \omega_0^2 m$. The other end of the spring is fixed and the body is set into motion about the fixed point with angular momentum per unit mass J and total mechanical energy per unit mass E.

1. Using as a guide the planetary Kepler problem from lectures, show that the body describes an orbit given by
$$\frac{1}{r^2} = \frac{E}{J^2}\left[1 + \left(1 - \frac{\omega_0^2 J^2}{E^2}\right)^{1/2} \cos 2\theta\right] \tag{2}$$
in the (r, θ) plane.

2. What shape is this locus? Draw a labelled diagram in the (x, y) plane.

3. What condition must be satisfied by ω_0, J, and E in order for the orbit to be bound?

4. Again using lectures as a guide, derive an effective potential per unit mass, $V_{\text{eff}}(r)$, for the body satisfying
$$\frac{d^2 r}{dt^2} = -\frac{dV_{\text{eff}}}{dr}\ . \tag{3}$$
Interpret your answer to the previous question in terms of this effective potential.

Now imagine a planetary system in which the force of gravity is spring-like, i.e. it is attractive and **directly** (not inversely!) proportional to the square of the separation between two bodies. (Needless to say, this is not a real planetary system, but it is very important in other contexts, as you will see later in the course.)

5. Derive the equivalent of Kepler's Third Law for this planetary system. What is special about your result?

6. What would be the motion of Mars in the sky as seen from Earth if the Earth and Mars were in this planetary system?

3 Copernicus versus Ptolemy

The planet Mars is observed to move slowly eastward relative to the fixed stars from night to night. Occasionally, however, it appears to double back on its path, moving westward for a few months. This is called a retrograde loop.

Consider the **heliocentric** model of Copernicus. Assume, by way of approximation, that Mars and Earth describe circular orbits, that these orbits are coplanar, and that Mars is in opposition at time $t = 0$, i.e. the Sun, Earth and Mars lie along a line, in that order, at $t = 0$.

Take $R_{\mathrm{E}} = 1\,\mathrm{AU}$, $\Omega_{\mathrm{E}} = 2\pi\,\mathrm{rad\,yr^{-1}}$, and $R_{\mathrm{M}} = 1.52\,\mathrm{AU}$.

1. What is Ω_{M}?

2. Carefully (and roughly to scale, if possible) draw a diagram of Copernicus's heliocentric system. By marking on it several sequential positions of Mars and Earth, show that Mars, as viewed from Earth, executes a retrograde loop against the background of the fixed stars in a time interval about $t = 0$.

3. Estimate approximately how long the retrograde loop lasts, as viewed from Earth.

4. Calculate the time that elapses between successive retrograde loops.

5. Calculate the angular separation between the locations of successive retrograde loops, as viewed from Earth. What would be the angular separation if Mars described a different orbit with $\Omega_{\mathrm{M}} = \pi\,\mathrm{rad\,yr^{-1}}$? Comment.

Ptolemy devised a **geocentric** explanation of retrograde loops involving epicycles.

6. Carefully (and roughly to scale, if possible) draw a diagram of Ptolemy's geocentric system. By marking on it several sequential positions of Mars and Earth, show how the geocentric system leads to retrograde loops.

7. What features of real planetary motion (as opposed to the idealised motion considered here) did Ptolemy's model fail to explain economically?

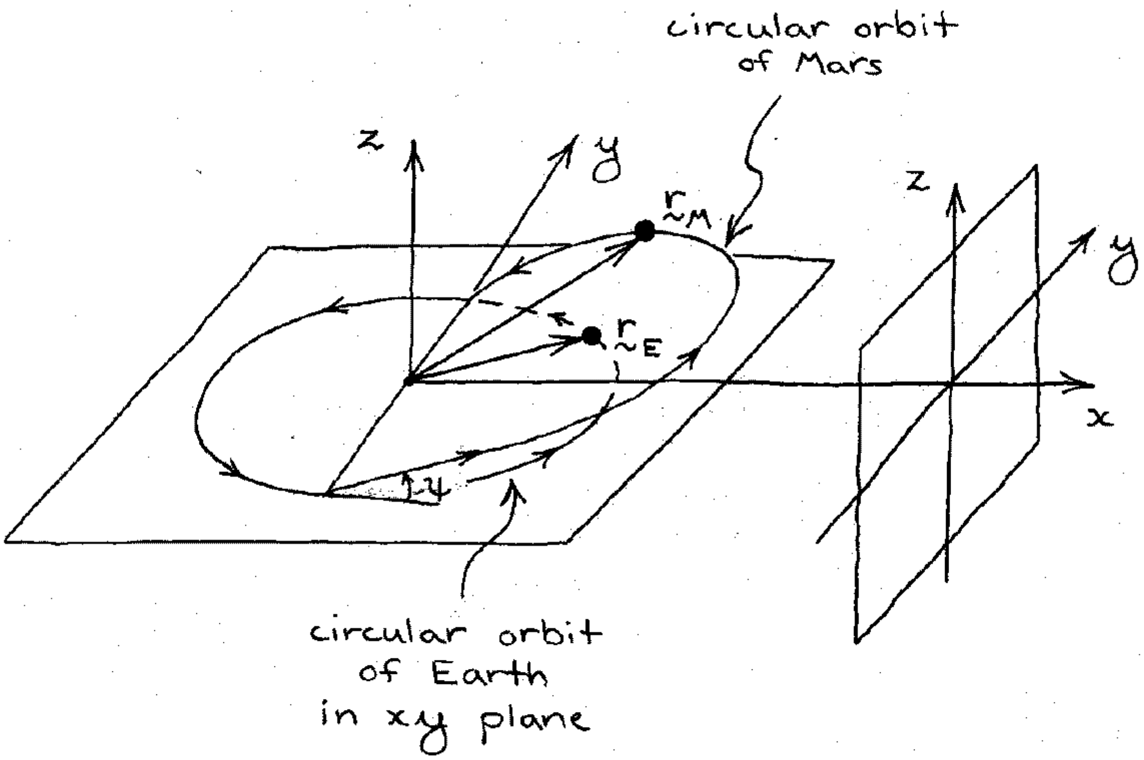


Figure 1:

In reality, the orbit of Mars is inclined with respect to the plane of the Earth's orbit (the ecliptic) by an angle $\psi = 1.85°$. This is shown in Fig. 1.

8. Write down $\mathbf{r}_{\mathrm{M}}(t)$ and $\mathbf{r}_{\mathrm{E}}(t)$.

9. What is the relative position of Mars as seen from Earth?

10. Project this relative position onto the y-z plane of the sky for $\Omega_{\mathrm{E}}t = -5\pi/8$, $-\pi/2$, $-3\pi/8$, $-\pi/4$, $-\pi/8$, 0, $\pi/8$, $\pi/4$, $3\pi/8$, $\pi/2$, $5\pi/8$. Graph the result neatly on ruled axes, appropriately scaled. You should get a retrograde loop.

4 Synodic versus Sidereal Periods

1. Explain, in your own words, the difference between synodic and sidereal periods.

Venus cycles through a complete set of phases every 584 d.

2. Draw the complete set of phases.

3. What is the synodic period of Venus? Its sidereal period?

Mars cycles through a complete set of phases every 780 d.

4. Draw the complete set of phases.
5. What is the synodic period of Mars? Its sidereal period?

Now imagine you are observing the Earth from Mars, instead of the other way around.

6. What would be the synodic period of the Earth? Its sidereal period?

5 Newton's and Gauss's Laws of Gravitation

Consider a spherical shell of material of uniform density, with radius R and total mass M.

1. Calculate the acceleration due to gravity **inside** the shell using Gauss's Law.
2. Redo the calculation using Newton's point-mass formula.
3. Calculate the acceleration due to gravity **outside** the shell using Gauss's Law.